On a Game of Roulette

A student analyses the Baron's last puzzle.

You will recall that the Baron proposed two games at a roulette wheel, both for a stake of one coin. In the first Sir R----- was to spin the wheel three times and mark the point closest to him after each spin. If the triangle thus formed enclosed the centre of the wheel he should have received a prize of four coins. In the second he should have won six and one half coins if a point chosen at random upon the face of the wheel was inside said triangle.

When figuring Sir R-----'s expected winnings in these games, we shall make lighter work of it if we recognise the puzzles' symmetry under rotation; after Sir R----- has chosen his three points we can spin the wheel again to reveal another, equally likely, outcome. When I mentioned this to the Baron it seemed to me that it upset him a little, although I can fathom no reason as to why it should have done so.

We can further exploit symmetry under reflection; looking at the wheel in a mirror also reveals an equally likely outcome.

Now, in the first game, these observations mean that we can assume that the first point is always at the top of the wheel, at twelve of the clock as it were, since we are free to rotate the wheel until this is so. We can further assume that the second point lies to its right, clockwise between the top and the bottom of the wheel since we can view the wheel through a mirror if it does not. The third point will, after these manipulations, still lie upon some random spot upon the edge of the wheel.

If we draw a line from the second point through the centre of the wheel we shall discover a fourth point upon its edge. It is plain to see that the third point must lie clockwise between the bottom of the wheel and this fourth point if the triangle is to enclose the centre of the wheel.

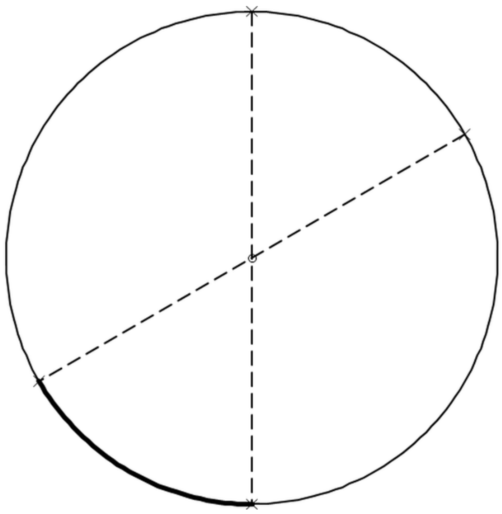


Figure 1

Figure 1 shows the set of winning third points marked in bold.

If the angle between the lines from the centre of the wheel to the first and second points is θ, then the probability that the triangle will enclose the centre is equal to

$$\frac{\theta}{2\pi}$$

The expected winnings of the game are equal to the average of this probability over all such θ multiplied by the prize of four coins, given by

$$\frac{4}{\pi} \times \int_0^\pi \frac{\theta}{2\pi} d\theta = \frac{4}{\pi} \times \left[\frac{\theta^2}{4\pi}\right]_0^\pi = 1$$

Since this is equal to Sir R-----'s stake I should have advised him that it was a fair game and he should have no compunction in playing if he so wished.

The winnings Sir R----- might expect from Baron's second game are a little more difficult to reckon. If we were to try the same trick that we used for the first game we should soon find ourselves tied up in knots.

We might try figuring the outcome of a number of games using pencil and paper, but if we do so we had better take care that the points on the face of the wheel are chosen with uniform probability; that the probability of a chosen point lying within a given region is some constant multiple of the area of that region.

When I explained this to the Baron his temper worsened considerably and he turned on his heel and left. I must confess that I remain utterly ignorant of how I might have offended him!

Now, it is not sufficient to pick a random angle and distance from the centre of the wheel when choosing points since this will concentrate the points at the centre of the wheel. A superior scheme by far is to pick points at random from within a square that surrounds the wheel and ignore those that do not lie upon its face (Listing 1).

In playing some few hundred games on paper, my fellow students and I found that the game seemed fair, but we were by no means certain.

Then it dawned upon me that, since the point is picked uniformly upon the face of the wheel, the probability of winning the game must be equal to the average area of the triangles divided by the area of the wheel.

To figure the average area of the triangles we can once again exploit the symmetries of the game. Specifically we rotate the wheel so that the first and second points lay either side of the line joining the top and bottom of the wheel and at the same height from the bottom. We may now assume that the third point will lie to the right of this line, since we can reflect the wheel if it does not (Figure 2).

To simplify matters, we shall use the radius of the wheel as our unit of length.

Now the area of such a triangle is equal to half of the length of the base multiplied by the height of the third point above, or depth below, it.

If the angle between the lines connecting the centre of the wheel and the top and rightmost point on the base is θ then, with a little trigonometry, we find that the length of the base is equal to

$$b = 2\sin\theta$$

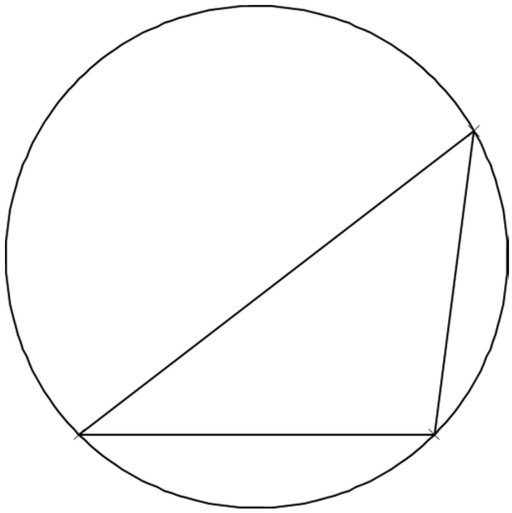
Listing 1

```
point
pick()
{
  double x, y;

  do
  {
    x = 2.0*double(rand())/
        (1.0+double(RAND_MAX)) - 1.0;
    y = 2.0*double(rand())/
        (1.0+double(RAND_MAX)) - 1.0;
  }
  while(x*x+y*y>=1.0);

  return point(x, y);
}
```

Figure 2



Similarly, if the angle so formed with the third point is α, then the height of the triangle is given by

$$h = \cos\alpha - \cos\theta$$

giving an area of

$$A = \sin\theta\cos\alpha - \sin\theta\cos\theta$$

If α is greater than θ then this area will be negative so we shall have to take care that we do not carelessly subtract the areas of such triangles from the average.

We shall do so by breaking the calculation of the average area into two parts. Firstly those triangles whose third point is above the base line and secondly those triangles whose third point is below it.

For a given θ, the average area of the triangles is thusly given by

$$\frac{1}{\pi} \times \left[\int_0^\theta (\sin\theta\cos\alpha - \sin\theta\cos\theta)d\alpha + \int_\theta^\pi -(\sin\theta\cos\alpha - \sin\theta\cos\theta)d\alpha\right]$$

$$= \frac{1}{\pi} \times [\sin\theta\sin\alpha - \alpha\sin\theta\cos\theta]_0^\theta - \frac{1}{\pi} \times [\sin\theta\sin\alpha - \alpha\sin\theta\cos\theta]_\theta^\pi$$

$$= \frac{\sin^2\theta - \theta\sin\theta\cos\theta}{\pi} - \frac{-\sin^2\theta - (\pi - \theta)\sin\theta\cos\theta}{\pi}$$

$$= 2\frac{\sin^2\theta - \theta\sin\theta\cos\theta}{\pi} + \sin\theta\cos\theta$$

To calculate the average area of any triangle we must perform a similar exercise upon this result.

$$\frac{2}{\pi^2}\int_0^\pi \sin^2\theta d\theta - \frac{2}{\pi^2}\int_0^\pi \theta\sin\theta\cos\theta d\theta + \frac{1}{\pi}\int_0^\pi \sin\theta\cos\theta d\theta$$

The third term is the simplest to figure since

$$\frac{d}{d\theta}\sin^2\theta = 2\sin\theta\cos\theta$$

Since $\sin\theta$ is equal to zero when θ is equal to both zero and π, this term is simply zero.

To figure the second term, we must use a technique known as integration by parts which states

$$\int u\frac{dv}{dx}dx = [uv] - \int v\frac{du}{dx}dx$$

If we take u to be θ, this yields

$$\int_0^\pi \theta\sin\theta\cos\theta d\theta = \frac{1}{2}[\theta\sin^2\theta]_0^\pi - \frac{1}{2}\int_0^\pi \sin^2\theta d\theta$$

The first of these terms is also zero, so the average area of the triangles must equal

$$\frac{3}{\pi^2}\int_0^\pi \sin^2\theta d\theta$$

Using integration by parts again, with both u and v equal to $\sin\theta$, we have

$$\int_0^\pi \sin^2\theta d\theta = [-\sin\theta\cos\theta]_0^\pi - \int_0^\pi -\cos^2\theta d\theta = \int_0^\pi \cos^2\theta d\theta$$

Adding the integral of the squared sine to both sides of this equation yields

$$2\int_0^\pi \sin^2\theta d\theta = \int_0^\pi \cos^2\theta d\theta + \int_0^\pi \sin^2\theta d\theta = \int_0^\pi (\cos^2\theta + \sin^2\theta)d\theta$$

and hence

$$\int_0^\pi \sin^2\theta d\theta = \frac{1}{2}\int_0^\pi (\cos^2\theta + \sin^2\theta)d\theta = \frac{1}{2}\int_0^\pi 1 d\theta = \frac{\pi}{2}$$

The average area of the triangles is therefore

$$\frac{3}{\pi^2} \times \frac{\pi}{2} = \frac{3}{2\pi}$$

Dividing this by the area of the wheel, which is trivially equal to π in the units we have adopted, and multiplying by the prize yields the expected winnings of this game

$$\frac{3}{2\pi} \times \frac{1}{\pi} \times 6\frac{1}{2} = \frac{39}{4\pi^2} < \frac{99}{100}$$

The game is consequently slightly biased in the Baron's favour and I could not in good conscience have advised Sir R----- to play.

Whilst my fellow students and I were considering the Baron's games we came to wonder what might make a fair prize if the point were chosen on the face of the wheel *before* the wager was made.

Considering the symmetry under rotation, it was readily apparent to us that the size of such bounty should depend only upon the distance between the point and the centre of the wheel.

By way of a careful approximation I found that the probability of winning such a wager was always within roughly one chance in a hundred of

$$\frac{\arccos r + r(1-r)}{2\pi}$$

where arccos is the inverse function of the cosine and r is the distance of the point from the centre of the wheel using the radius of the wheel as the unit of length. Unfortunately not one of us has managed to bring this puzzle to a tidy conclusion. I hardly need add that the fact that so simply stated a wager has entirely evaded our most strenuous efforts has been a source of no little frustration to me and my fellows. Figure 3 shows the error in the approximation against r. ■

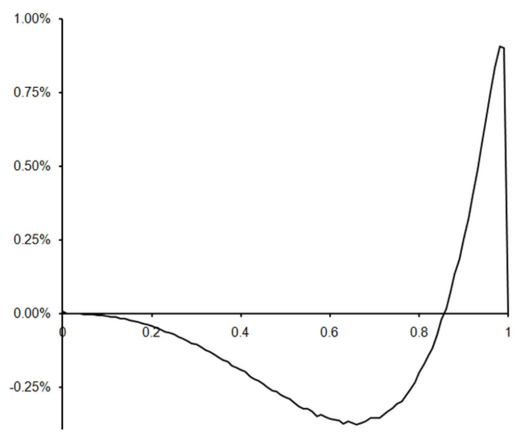


Figure 3